Ch.1 Distributions of Functions of Random Variables

Let $X_1, X_2, \dots, X_n$ be n random variables $r.v.'s$ that have the joint probability density function (j. $p.d.f.$) $f(x_1, x_2, \dots, x_n)$. These $r.v.'s$ may or may not be stochastically independent. Let $Y = U(X_1, X_2, \dots, X_n)$. Once the $p.d.f.$ $f(x_1, x_2, \dots, x_n)$ is given, we can find the $p.d.f.$ of Y.

Definition 1: A function of one or more $r.v.'s$ that does not depend upon any parameter is called a statistic .

Example: If the $r.v.'s$ X_i, $i = 1,2,\dots,n$ are mutually stochastically independent and each X_i has the same $p.d.f.$

$$f(x) = \begin{cases} p^x(1-p)^{1-x}, & x = 0,1; 0 < p < 1 \\ 0 & , \quad e.w \end{cases},$$

and if $Y = \sum_{i=1}^{n} X_i$ is a statistic.

Example: If the $r.v.$ $X \sim N(\mu, \sigma^2)$, and if $Y = \frac{X-\mu}{\sigma}$ is not statistic unless μ and σ^2 are known numbers.

Note: Although a statistic does not depend upon any unknown parameter, but the distribution of a statistic may very well depend upon the unknown parameters.

Definition 2: Let $X_1, X_2, \dots, X_n$ be n mutually stochastically independent $r.v.'s$ each of which has the same $p.d.f.$ $f(x)$, that is the $p.d.f.'s$ of $X_1, X_2, \dots, X_n$ are $f(x_1), f(x_2), \dots, f(x_n)$ respectively , so that the joint $p.d.f.$

$$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2)\dots f(x_n) = \prod_{i=1}^{n} f(x_i).$$

The $r.v.'s$ $X_1, X_2, \dots, X_n$ are then said to constitute a random sample of size n from a distribution that has $p.d.f.$ $f(x)$.

Note: random sample Ξ independent identically distribution

i.e.

$r.s. \Xi$ ii$d$.

Definition 3: Let $X_1, X_2, \dots, X_n \overset{iid}{\sim} f(x)$. The statistic

$$\overline{X} = \frac{X_1 + \cdots + X_n}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is called the mean of the random sample . And the statistic

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2,$$

is called the variance of the random sample.

Note : Let $X_1, X_2, \dots, X_n \sim f(x)$. Let $Y = U(X_1, X_2, \dots, X_n)$, distribution function of Y: $G(Y) = P_r(Y \le y) = P_r, U(X_1, X_2, \dots, X_n) \le y$-.

Discrete Probability Distribution and its Related

Introduction

For given random variables, say $X_1, X_2, \dots, X_n$ denoted (n) $r.v.^{'s}$ having the joint $p.d.$f. $f(x_1, x_2, \dots, x_n)$ and that

$$Y_1 = U_1(X_1, \dots, X_n),\ Y_2 = U_2(X_1, \dots, X_n), \dots,\ Y_k = U_k(X_1, \dots, X_n),$$

we want in general to find the joint distribution of $Y_1, Y_2, \dots, Y_k$, where $k \le n$ as well as the marginal distribution of any combination of $Y_1, Y_2, \dots, Y_k$. There are three techniques are called:

(1) The cumulative distribution function techniques .

(2) The moment generating function techniques.

(3) The transformation techniques.

(1) The cumulative distribution function techniques

Let the joint probability density function of $r.vs$ $X_1, X_2, \dots, X_n$ is given as $f(x_1, x_2, \dots, x_n)$, then the j. $p.d.f.$ of $r.v.'s$ $Y_1, Y_2, \dots, Y_k$ can be determined, where

$Y_j = U_j(X_1, \dots, X_n), j = 1,2, \dots, k, k \le n$ by using $c.d.f.$ technique , then

$$F(y_1, \dots, y_k) = P_r(Y_1 \le y_1, Y_2 \le y_2, \dots, Y_k \le y_k)$$

$$= P_r\{U_1(X_1, \dots, X_n) \le y_1, \dots, U_k(X_1, \dots, X_n) \le y_k\}.$$

If Y is univariate random variable , then

$F(y) = P_r(Y \le y) = P_r\{U(X_1, \dots, X_n) \le y\}$, then , the $p.d.f.$ of Y is

$$f(y) = \frac{\partial F(y)}{\partial y},$$

and the j. $p.d.f.$ of $Y_1, Y_2, \dots, Y_k$ is $f(y_1, y_2, \dots, y_k) = \frac{\partial F(y_1, \dots, y_k)}{\partial y_1 \partial y_2 \dots \partial y_k}$.

Example: Let X be $r.v.$ having $p.m.f.$ $f(x) = \frac{x}{6}, x = 1,2,3$, $zero$, $otherwise$. Find the distribution of $Y = X^2$ by using $c.d.f.$ technique .

Sol.

$F(y) = P_r(Y \le y) = P_r(X^2 \le y) = P_r(-\sqrt{y} \le X \le \sqrt{y}) = P_r(X \le \sqrt{y})$

$$= \sum_{x=1}^{\sqrt{y}} \frac{x}{6} = \frac{1}{6} \sum_{x=1}^{\sqrt{y}} x,$$

<u>**Recall that**</u> , by using the summation of natural number law for $\sum_{i=1}^{\sqrt{y}} x_i$, then

$F(y) = \frac{1}{6} \frac{\sqrt{y}(\sqrt{y}+1)}{2} = \frac{\sqrt{y}(\sqrt{y}+1)}{12}$.

$A = \{x : x \in R, x = 1,2,3\}, B = \{y : y \in R, y = 1,4,9\}.$

$$\therefore F(y) = \begin{cases} 0\,, & y < 1 \\ \frac{1}{6}\,, & 1 \le y < 4 \\ \frac{1}{2}, & 4 \le y < 9 \\ 1\,, & y \ge 9 \end{cases},$$

$$\rightarrow f(y) = \begin{cases} \frac{1}{6}, & y = 1 \\ \frac{1}{3}, & y = 4 \\ \frac{1}{2}\,, & y = 9 \\ 0\,, & 0.w \end{cases}.$$

(2) The Moment Generating Function

The another method for determining the distribution of function of $r.v.'^s$ which we shall find to be particularly useful in certain instances.

For given $r.v.'^s$ $X_1, X_2, \dots, X_n$ with given density $f(x_1, \dots, x_n)$ and given functions $g_1(X_1, \dots, X_n), g_2(X_1, \dots, X_n), \dots, g_k(X_1, \dots, X_n)$. To find the joint distribution of $Y_1 = g_1(X_1, \dots, X_n), \dots, Y_k = g_k(X_1, \dots, X_n)$. Now the joint moment generating function of $Y_1, \dots, Y_k$ if it exists is:

$$M_{Y_1,\dots,Y_k}(t_1, \dots, t_k) = E[e^{t_1Y_1+t_2Y_2+\dots+t_kY_k}]$$

$$= \sum_{x_1} \dots \sum_{x_n} e^{t_1Y_1+t_2Y_2+\dots+t_kY_k} f(x_1, \dots, x_n)$$

$$= \sum_{x_1} \dots \sum_{x_n} e^{t_1 g_1(x_1,\dots,x_n)+\dots+t_k g_k(x_1,\dots,x_k)} f(x_1, \dots, x_n).$$

Theorem: If $X_1, X_2, \dots, X_n$ are independent $r.vs$ and the $m.g.f.$ of each exists for all $-h < t < h$, for some $h > 0$, let $Y = \sum_{i=1}^{n} X_i$, then $M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t)$.

Proof

$$M_{\mathrm{F}}(t) = E(e^{t\mathrm{F}}) = E(e^{t\sum_{\mathrm{i}=1}^{n}\mathrm{K_i}}) = E,e^{t(\mathrm{K}_1+\mathrm{K}_2+\cdots+\mathrm{K}_n)}-$$

$$= E,e^{t\mathrm{K}_1+t\mathrm{K}_2+\cdots+t\mathrm{K}_n}- = E,e^{t\mathrm{K}_1}.\, e^{t\mathrm{K}_2} \ldots .\, e^{t\mathrm{K}_n}-,$$

but $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ are independent , then

$M_{\mathrm{F}}(t) = E,e^{t\mathrm{K}_1}-E,e^{t\mathrm{K}_2}- \ldots E,e^{t\mathrm{K}_n}- = \prod_{\mathrm{i}=1}^{n} M_{\mathrm{K_i}}(t).$

Example: Let $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n$ are independent Bernoulli $r.vs$, find the distribution of $Y = \sum_{\mathrm{i}=1}^{n} \mathrm{X_i}$.

Sol.

$$M_{\mathrm{F}}(t) = E(e^{t\mathrm{F}}) = E(e^{t\sum_{\mathrm{i}=1}^{n}\mathrm{K_i}})$$

$\therefore \mathrm{X_i} \sim Ber(p), \mathrm{i} = 1,2, \ldots, n$

$\rightarrow M_{\mathrm{K_i}}(t) = (q + pe^t)$

$\therefore M_{\mathrm{F}}(t) = (q + pe^t)(q + pe^t) \ldots (q + pe^t) = (q + pe^t)^n$

$$\therefore Y \sim b(n, p)$$

Note: we can say, if $X_1, X_2, \dots, X_n$ be independent $r.v.'^s$ and we wish to know the distribution of $Y = \sum_{i=1}^{n} X_i$:

a. $X_i \sim b(m,p), i = 1,2,\dots,n \rightarrow Y = \sum_{i=1}^{n} X_i \sim b(nm,p)$.

b. $X_i \sim Po(\lambda), i = 1,2,\dots,n \rightarrow Y = \sum_{i=1}^{n} X_i \sim Po(n\lambda)$.

c. $X_i \sim geom.(p), i = 1,2,\dots,n \rightarrow Y = \sum_{i=1}^{n} X_i \sim Nb(n,q)$, where $r = n$.

(3) The transformation technique (change of variable technique)

(a) One- dimensional space

A $r.v.$ X may be transformed by some functions $g(x)$ to define a new $r.v.Y$. The density of Y, will be determined by the transformation $g(x)$ togather with the density $f(x)$ of X . Sometimes this method called the change of variable technique. The transformation of variables of discrete type is:

If X is a discrete $r.v.$ with mass points $x_1, x_2, \dots, x_n$, then the distribution of $Y = g(X)$ is determined directly by the laws of probability. Let $f(x)$ be the $p.m.f.$, the sample space of X is A which $f(x) > 0$, let $g(y)$ be one - to – one transformation , that maps A onto B , the $p.m.f.$ of $Y = g(X)$ is given

$$g(y) = \begin{cases} f[w(y)] & , y \in B \\ 0 & , o.w. \end{cases} ,$$

where $X = w(Y)$ is the inverse of $Y = g(X)$.

Example: Let X have a Poisson $p.d.f.$, find the distribution of $Y = 4X$ using transformation technique.

Sol.

$$f(x;\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^{x}}{x!} , & x = 0,1,2,\ldots , \\ 0 , & o.w. \end{cases}$$

$$A = \{x: x \in R, x = 0,1,2,\ldots,\infty\}, f(x) > 0$$

$$B = \{y: y \in R, y = 0,4,8,\ldots,\infty\}, g(y) > 0$$

$$g(y) = P_r(Y = y) = P_r\left(X = \frac{y}{4}\right)$$

$$\rightarrow g(y) = \begin{cases} \frac{e^{-\lambda}\lambda^{\frac{y}{4}}}{\left(\frac{y}{4}\right)!} , & y = 0,4,8,\ldots \\ 0 , & o.w. \end{cases} .$$

Example: Suppose that X be binomial distribution with $n = 3$, $p = \frac{2}{3}$, where $x = 0,1,2,3$ find the distribution of $Y = X^2$ by using one – to – one transformation.

Sol.

$$f(x;n,p) = \begin{cases} \binom{3}{x}\left(\frac{2}{3}\right)^{x}\left(\frac{1}{3}\right)^{3-x} , & x = 0,1,2,3 \\ 0 , & o.w. \end{cases} .$$

$A = \{x: x \in R, x = 0,1,2,3\}, f(x) > 0$, (sample space of X)

$B = \{y: y \in R, y = 0,1,4,9\}, g(y) > 0$, (sample space of Y)

$$g(y) = P_r(Y = y) = P_r(X^2 = y) = P_r(X = \pm\sqrt{y}) = P_r(X = \sqrt{y})$$

$$g(y) = \begin{cases} \binom{3}{\sqrt{y}} \left(\frac{2}{3}\right)^{\sqrt{y}} \left(\frac{1}{3}\right)^{3-\sqrt{y}} & , y = 0,1,4,9 \\ 0 & , \quad o.w. \end{cases}.$$

Example : Let $f(-1) = \frac{1}{3}$, $f(0) = \frac{1}{2}$, and $f(1) = \frac{1}{6}$. Find the $p.m.f.$ of $Y = X^2$ by using transformation method.

Sol.

$$\therefore y = x^2 \rightarrow x \pm \sqrt{y}$$

x	-1	0	1
y	1	0	1

$A = *x: x \in R, x = -1,0,1+, f(x) > 0,$

$B = *y: y \in R, y = 0,1+, g(y) > 0,$

$g(y = 0) = f(x = 0) = \frac{1}{2},$

$g(y = 1) = p\,(x = -1) + p(x = 1) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2},$

$$\therefore g(y) = \begin{cases} \frac{1}{2} & , \quad y = 0 \\ \frac{1}{2} & , \quad y = 1 \\ 0 & , \quad o.w. \end{cases}.$$

(b)Two – dimensional space

Let $f(x_1, x_2)$ be the j. $p.\ d.$ f. of two discrete type $r.\ v.'^s$ X_1 and X_2 with sample space A which $f(x_1, x_2) > 0$. Let define the one – to – one transformation that map A onto B, the j. $p.d.$f. of $Y_1 = U_1(X_1, X_2)$ and $Y_2 = U_2(X_1, X_2)$ is given by

$$g(y_1, y_2) = \begin{cases} f[w_1(y_1, y_2), w_2(y_1, y_2)] , & (y_1, y_2) \in B \\ 0 , & o.w. \end{cases},$$

where $x_1 = w_1(y_1, y_2)$ and $x_2 = w_2(y_1, y_2)$ are the single valued inverse of $Y_1 = U_1(X_1, X_2)$ and $Y_2 = U_2(X_1, X_2)$. From this j. $p.\ m.$ f. we may obtain the marginal of Y_1 and Y_2 we note that one – to – one transformation means that each point in sample space A of $r.v.$ X correspond one to one point in sample space B of $r.v.Y$ and conversely.

Example: Let X_1 and X_2 be two stochastically independent $r.\ vs$ that have Poisson distribution with means μ_1 and μ_2 respectively . If $Y = X_1 + X_2$ find the $p.d.$f. of Y.

Sol.

$$f(x_1, x_2, \mu_1, \mu_2) = f(x_1, \mu_1) f(x_2, \mu_2)$$

$$= \frac{e^{-\mu_1}\mu_1^{x_1}}{x_1!} \frac{e^{-\mu_2}\mu_2^{x_2}}{x_2!} = \frac{e^{-\mu_1-\mu_2}\mu_1^{x_1}\mu_2^{x_2}}{x_1!\, x_2!}$$

$$y_1 = x_1 + x_2 \text{ , let } y_2 = x_2 \rightarrow x_1 = y_1 - y_2 \text{ and } x_2 = y_2$$

$$\therefore A = \{(x_1, x_2): x_1 = 0,1,2, \dots, \infty, x_2 = 0,1,2, \dots, \infty\}, f(x_1, x_2) > 0$$

$$B = \{(y_1, y_2): y_1 = 0,1, \dots, \infty, y_2 = 0,1, \dots, \infty\}, g(y_1, y_2) > 0$$

$$\therefore g(y_1, y_2) = \begin{cases} \dfrac{e^{-\mu_1-\mu_2}\mu_1^{y_1-y_2}\mu_2^{y_2}}{(y_1-y_2)!\,y_2!} & , \quad y_1 = 0,1,2,\dots, \\ & \quad y_2 = 0,1,2,\dots, y_1 \\ 0 & , \quad o.w. \end{cases}$$

$$g(y_1) = \sum_{y_2} g(y_1, y_2) = \sum_{y_2=0}^{y_1} \frac{e^{-\mu_1-\mu_2}\mu_1^{y_1-y_2}\mu_2^{y_2}}{(y_1-y_2)!\,y_2!} ,$$

we multiple and divide the term inside the summation by $y_1!$, then

$$g(y_1) = \frac{e^{-\mu_1-\mu_2}}{y_1!} \sum_{y_2=0}^{y_1} \frac{y_1!}{y_2!\,(y_1-y_2)!} \mu_2^{y_2} \mu_1^{y_1-y_2}$$

$$\rightarrow g(y_1) = \frac{e^{-\mu_1-\mu_2}}{y_1!} \sum_{y_2=0}^{y_1} \binom{y_1}{y_2} \mu_2^{y_2} \mu_1^{y_1-y_2} .$$

Recall that: $(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$, then

$$g(y_1) = \frac{e^{-(\mu_1+\mu_2)}(\mu_1+\mu_2)^{y_1}}{y_1!} \rightarrow Y_1 \sim Po(\mu_1+\mu_2)$$

$$g(y_1) = \begin{cases} \dfrac{e^{-(\mu_1+\mu_2)}(\mu_1+\mu_2)^{y_1}}{y_1!} & , \quad y_1 = 0,1,2,\dots \\ 0 & , \quad o.w. \end{cases}.$$

Continuous Probability Distribution and its Related

(1) The cumulative distribution function technique

By using the same way of discrete $r.v.'^{s}$ can be using $c.d.f.$ technique.

Example:

Let X_1 and X_2 be a random sample of size 2 from the exponential distribution with $\lambda = 1$. Find the probability distribution of

$$Y = X_1 + X_2 \text{ .}$$

Sol.

Since $X_1 \sim Ex(\lambda)$ or $X_1 \sim Exp(1)$ and $X_2 \sim Exp(1)$

$$\therefore f(x_1) = \begin{cases} e^{-x_1}, & x_1 > 0 \\ 0, & o.w. \end{cases}, \text{ and } f(x_2) = \begin{cases} e^{-x_2}, & x_2 > 0 \\ 0, & o.w. \end{cases}.$$

Since X_1 and X_2 are independent

$$\rightarrow f(x_1, x_2) = f(x_1)f(x_2) = e^{-x_1} e^{-x_2}$$

$$= \begin{cases} e^{-x_1-x_2}, & x_1 > 0, x_2 > 0 \\ 0, & o.w. \end{cases}.$$

$(y) = P_r(Y \leq y) = P_r(X_1 + X_2 \leq y) = \iint_A \quad f(x_1, x_2)dx_2dx_1,$

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where $A = \{(x_1, x_2): x_1 > 0\ , x_2 > 0, x_1 + x_2 \leq y\}$

$$\rightarrow A = \{(x_1, x_2): 0 < x_1 < \ ,0 < x_2 < y - x_1\}$$

$$\rightarrow (y) = \int_0^y \int_0^{y-x_1} \quad e^{-x1-x_2} dx_2dx_1 = \int_0^y e^{-x1} [\int_0^{y-x_1} \quad e^{-x2} dx_2]\, dx_1$$

$$= -\int_0^y e^{-x1} [e^{-y+x1} - 1]x_1 = \int_0^y [e^{-x1} - e^{-y}]dx_1$$

$$= 1 - e^{-y} - ye^{-y}$$

$B = \{y: y \in R, y > 0\}$, $f(y) = F'(y) = \frac{6F}{6y} = ye^{-y}$

$\therefore f(y) = \{\begin{matrix} ye^{-}, & y > 0 \\ 0, & o.w. \end{matrix}$

$$\therefore Y \sim Gamm(2, 1).$$

(2) Moment generating function technique

Also as in discrete type can be use the $m.g.f.$ and by same way .

Example:

Assume that $X_1, X_2, \ldots, X_n$ be independent exponential distribution.
Find the $p.d.f.$ of $Y = \sum_{i=1}^{n} X_i$

Sol.

$$M_F(t) = E(e^{tF}) = E(e^{t(K_1+K_2+\cdots+K_n)}) = \mathrm{G}_{i=1}^{n} M_{K_i}(t)$$

$$\therefore X_i \sim Ex(\beta), ✯\, i = 1,2,\ldots,n$$

$$M_{K_i}(t) = \frac{1}{1-\beta t}$$

$$\therefore M_F(t) = \mathrm{G}_{i=1}^{n} \left(\frac{1}{1-\beta t}\right) = \left(\frac{1}{1-\beta t}\right)^{n}$$

$$\therefore Y \sim Gamma(n, \mathrm{Q})$$

$$\therefore g(y) = \begin{cases} \frac{1}{\Gamma(n)\beta^{n}} y^{n-1} e^{-\frac{y}{\beta}}, & y > 0 \\ o, & o.w. \end{cases}$$

Theorem: Let $X_1, X_2, \dots, X_n$ be independent random variables such that $X_i \sim N(\mu_i, \sigma_i^2)$, ✯ $i = 1,2,\dots,n$. Let $Y = \sum_{i=1}^{n} k_i X_i$, $k_i \in R$, $i = 1,2,\dots,n$. Then $Y \sim (\mu_F, \sigma_F^2)$, where $\mu_F = \sum_{i=1}^{n} k_i \mu_i$, $\sigma_F^2 = \sum_{i=1}^{n} k_i^2 \sigma_i^2$.

Proof:

$$\therefore M_F(t) = (e^{tF}) = E(e^{t \sum_{i=1}^{n} k_i K_i}) = (e^{tk_1 K_1 + tk_2 K_2 + \cdots + tk_n K_n})$$

$\therefore X_1, X_2, \dots, X_n$ are independent

$$\therefore M_F(t) = E(e^{tk_1 K_1}) E(e^{tk_2 K_2}) \dots E(e^{tk_n K_n})$$

Since X_i are independent ✯ $i = 1,2,\dots,n$ and $X_i \sim N(\mu_i, \sigma_i^2)$

$\rightarrow M_{K_i}(t) = e^{\mu_i t + \frac{1}{2}\sigma_i^2}$, ✯ $i = 1,2,\dots,n$.

$$\therefore M_F(t) = e^{k_1 \mu_1 t + \frac{1}{2} k_1^2 \sigma_1^2 t^2} \dots e^{k_n \mu_n t + \frac{1}{2} k_n^2 \sigma_n^2 t^2}$$

$$= e^{(\sum_{i=1}^{n} k_i \mu_i) + \frac{1}{2} (\sum_{i=1}^{n} k_i^2 \sigma_i^2)^2}$$

$\rightarrow Y \sim (\mu_F, \sigma_F^2)$ or $Y \sim N(\sum_{i=1}^{n} k_i \mu_i, \sum_{i=1}^{n} k_i^2 \sigma_i^2)$

Theorem: Let $X_1, X_2, \dots, X_n$ be independent $r.v.'s$ such that $X_i \sim \chi^2(r_i)$, ✭i $= 1,2,\dots,n$. Then $Y = \sum_{i=1}^{n} X_i \sim \chi^2(\sum_{i=1}^{n} r_i)$.

Proof:

$$M_F(t) = (e^{tF}) = E(e^{t\sum_{i=1}^{n} K_i}) = (e^{tK_1 + tK_2 + \cdots + tK_n})$$

$\therefore X_1, X_2, \dots, X_n$ are independent

$\therefore M_F(t) = E(e^{tK_1})E(e^{tK_2}) \dots E(e^{tK_n})$.

Since $X_i \sim \chi^2(r_i)$, ✭i $= 1,2,\dots,n, \rightarrow M_{K_i}(t) = (\frac{1}{1-2t})^{\frac{r_i}{2}}$, ✭i $= 1,2,\dots,n$.

$$\therefore M_F(t) = (\frac{1}{1-2t})^{\frac{r_1}{2}} (\frac{1}{1-2t})^{\frac{r_2}{2}} \dots (\frac{1}{1-2t})^{\frac{r_n}{2}} = (\frac{1}{1-2t})^{\sum_{i=1}^{n} \frac{r_i}{2}}$$

$\rightarrow Y = \sum_{i=1}^{n} X_i \sim \chi^2(\sum_{i=1}^{n} r_i)$.

Theorem: Let $X_1, X_2, \dots, X_n \sim$ iid $N(\mu, \sigma^2)$. Then

$$Y = \sum_{i=1}^{n} (\frac{X_i - \mu}{\sigma})^2 \sim \chi^2(n)$$

Proof:

Since $X_i \sim (\mu, \sigma^2)$, let $W_i = \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(1)$

$$Y = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2\left(\sum_{i=1}^{n} 1 = n\right).$$

(3) Transformation of variables of the continuous type

(a) One – dimensional space

Let X be a random variable of the continuous type having $p.d.f.$ $f(x)$. Let A be the one – dimensional space where $f(x) > 0$. Consider the random variable $Y = U(X)$, where $y = (x)$ define a one – to – one transformation which maps the set A onto the set B. Let the inverse of $y = (x)$ be denoted by $x = w(y)$, and let the derivative $\frac{dx}{dy} = w'(y)$ be continuous and not vanish for all points in B. Then the $p.d.f.$ of the random variable $Y = U(X)$ is given by

$$g(y) = \begin{cases} f[w(y)]|w'(y)| & , y \in B \\ 0 & , o.w. \end{cases},$$

where $|w'(y)|$represent the absolute value of $w'(y)$. We shall refer to $\frac{dx}{dy} = w'(y)$ as the Jacobian of the transformation and denoted by J. So

$$(y) = \begin{cases} f[w(y)]|J| & , y \in B \\ 0 & , o.w. \end{cases}.$$

Example:

Let X be a $r.v.$ of the continuous type, having $p.d.f.$

$$f(x) = \begin{cases} 2 \,, 0 < x < 1 \\ 0 \,, \quad o.w. \end{cases}.$$

Define the $r.v.$ $Y = 8X^3$. Find $p.d.f.$ of Y.

Sol.

Since $y = 8x^3 \rightarrow x = \frac{1}{2}\sqrt[3]{y} \rightarrow \frac{d}{dy} = \frac{1}{6\sqrt[3]{y^2}} \therefore |J| = |\frac{1}{6\sqrt[3]{y^2}}|$,

$$(y) = f[w(y)]|J| = f(\frac{1}{2}\sqrt[3]{y})|\frac{1}{6\sqrt[3]{y^2}}|$$

$$= 2.\frac{1}{2}\sqrt[3]{y}\ \frac{1}{6\sqrt[3]{y^2}} = \frac{1}{6\sqrt[3]{y}}$$

$$g(y) = \begin{cases} \frac{1}{6\sqrt[3]{y}} , & 0 < y < 8 \\ 0 , & o.w. \end{cases}.$$

(b) Two – dimensional space

Let X_1 and X_2 be random variables of continuous type, having joint $p.\ d.\ f.$ $f(x_1,\ x_2$. Let A be the two – dimensional set in the $x_1x_2 -$ plane where $f(x_1,\ x_2) > 0$. Let $Y_1 = U_1(X_1,\ X_2)$ be a random variable whose $p.d.f.$ to be found. If $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$define a one – to – one transformation of A onto B in $y_1y_2 -$ plane, we can find the joint $p.d.f.$ of $Y_1 = U_1(X_1, X_2)$ and $Y_2 = U_2(X_1, X_2)$. So

$$(y_1\ \ , y_2) = \begin{cases} f[w_1(y_1, y_2), w_2(y_1, y_2)]|J| , & (y_1, y_2) \in B \\ 0 & , \quad o.w. \end{cases} ,$$

where , $J = \begin{vmatrix} \frac{6x_1}{6y_1} & \frac{6x_1}{6y_2} \\ \frac{6x_2}{6y_1} & \frac{6x_2}{6y_2} \end{vmatrix}$.

Example:

Let X_1 and X_2 be a random sample of size 2 from the distribution having $p.d.f.$

$$f(x) = \begin{cases} e^{-} & , 0 < x < \infty \\ 0 & , \quad e.w. \end{cases} .$$

Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$, we shall show that Y_1 and Y_2 are stochastically independent.

Sol.

Since the joint $p.d.f.$ of X_1 and X_2 is

$$f(x_1, x_2) = f(x_1)f(x_2) = \begin{cases} e^{-(x_1+x_2)} & , 0 < x_1 < \infty , 0 < x_2 < \infty \\ 0 & , \quad o.w. \end{cases}$$

$$\left. \begin{aligned} y_1 &= x_1 + x_2 \\ y_2 &= \frac{x_1}{x_1 + x_2} \end{aligned} \right\} \Rightarrow \begin{aligned} x_1 &= y_1 y_2 \\ x_2 &= y_1 - y_1 y_2 \end{aligned}$$

$$\therefore J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{vmatrix} = -y_1 y_2 - y_1 + y_1 y_2 = -y_1$$

$$\therefore |J| = |-y_1| = y_1$$

$$\therefore g(y_1, y_2) = \begin{cases} f[w_1(y_1,y_2), w_2(y_1,y_2)]|J|, & (y_1,y_2) \in B \\ 0, & o.w. \end{cases}$$

$$= \begin{cases} y_1 e^{-y_1}, & 0 < y_1 < \infty, 0 < y_2 < 1 \\ 0, & o.w. \end{cases}$$

$$\to (y_1) = \int_{y_2} g(y_1, y_2) dy_2 = \int_0^1 y_1 e^{-y_1} dy_2 = y_1 e^{-y_1} [y_2]_0^1$$

$$= \begin{cases} y_1 e^{-y_1}, & 0 < y_1 < \infty \\ 0, & o.w. \end{cases}$$

$$\to (y_2) = \int_0^\infty g(y_1, y_2) dy_1 = (2) = (2 - 1)$$

$$= \begin{cases} 1, & 0 < y_2 < 1 \\ 0, & o.w. \end{cases}.$$

Since $(y_1, y_2) = g(y_1) g(y_2) \Rightarrow y_1$ **and** y_2 **are independent.**

The t — distribution

Let W denote a random variable which is N(0,1); Let V denote a random variable which is $\chi^2(r)$; Let W and V be stochastically independent i.e.

$$W \sim N(0,1) \Rightarrow f(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2}, -\infty < w < \infty$$

$$V \sim \chi^2(r) \Rightarrow f(v) = \frac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}}, 0 < v < \infty$$

indep.

Define a new random variables T and U by writing $T = \frac{W}{\sqrt{\frac{V}{r}}}$, $U = V$. The change of variable technique will be used to find the p.d.f. $g_1(t)$ of T. The equations $T = \frac{w}{\sqrt{\frac{v}{r}}}$, $u = v$ define a one – to – one transformation which maps

$$A = \{(w, v): -\infty < w < \infty, 0 < v < \infty\}$$

onto

$$B = \{(t, u): -\infty < t < \infty, 0 < u < \infty\}.$$

Now $w = t\sqrt{\frac{u}{r}}$, $v = u$

$$\begin{vmatrix} \frac{\partial w}{\partial t} & \frac{\partial w}{\partial u} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{r}} & \frac{t}{2\sqrt{ur}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{r}} \Rightarrow |J| = \left|\sqrt{\frac{u}{r}}\right| = \sqrt{\frac{u}{r}} = \frac{\sqrt{u}}{\sqrt{r}}$$

$$\Rightarrow g(t, u) = f\left(t\sqrt{\frac{u}{r}}, u\right)|J|$$

$$\Rightarrow g(t,u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{t^2 u}{r}} \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}} \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi}\ \Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} u^{\frac{r}{2}-1} e^{-\frac{u}{2}(1+\frac{t^2}{r})} \frac{\sqrt{u}}{\sqrt{r}}$$

$$= \frac{1}{\sqrt{2\pi r}\ \Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} u^{\frac{r-1}{2}} e^{-\frac{u}{2}(1+\frac{t^2}{r})}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi r}\ \Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} u^{\frac{r+1}{2}-1} e^{-\frac{u}{2}(1+\frac{t^2}{r})}, & -\infty < t < \infty,\ 0 < u < \infty. \\ 0, & o.w. \end{cases}$$

The marginal p.d.f. of T is then

$$g_1(t) = \int_{-\infty}^{\infty} g(t,u)\,du = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi r}\ \Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} u^{\frac{r+1}{2}-1} e^{-\frac{u}{2}(1+\frac{t^2}{r})}\, du$$

In this integral let $z = \frac{u(1+\frac{t^2}{r})}{2}$, $dz = \frac{(1+\frac{t^2}{r})}{2}\, du \Rightarrow du = \frac{2}{1+\frac{t^2}{r}}$, $u = \frac{2z}{1+\frac{t^2}{r}}$

$$g_1(t) = \int_0^{\infty} \frac{1}{\sqrt{2\pi r}\ \Gamma(\frac{r}{2})\, 2^{\frac{r}{2}}} \left(\frac{2z}{1+\frac{t^2}{r}}\right)^{\frac{r+1}{2}-1} e^{-z} \left(\frac{2}{1+\frac{t^2}{r}}\right) dz$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi r}\;\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} \frac{2^{\frac{r+1}{2}-1} z^{\frac{r+1}{2}-1}}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}-1}} e^{-z} \left(\frac{2}{1+\frac{t^2}{r}}\right) dz$$

$$= \frac{1}{\sqrt{\pi r}\;\Gamma\left(\frac{r}{2}\right)} \frac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}} \int_0^\infty z^{\frac{r+1}{2}-1} e^{-z}\, dz$$

$$= \begin{cases} \dfrac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\;\Gamma\left(\frac{r}{2}\right)} \dfrac{1}{\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}, & -\infty < t < \infty \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore T = \frac{W}{\sqrt{\frac{V}{r}}} \sim t(r)$$

i.e. T has t —distribution with r degrees of freedom (d. f.)

$$T = \frac{W \sim N(0,1)}{\sqrt{\frac{\chi^2(r)}{r = d.f.}}} \sim t(r)$$

Note:

The c.d.f. $G(t) = P_r(T \le t) = \int_{-\infty}^{t} g_1(w)dw$ has been tabulated.

The F — distribution

Consider two stochastically independent chi-square random variables U and V having r_1 and r_2 degrees of freedom, respectively, i.e.

$$\left.\begin{aligned} U \sim \chi^2(r_1) &\Rightarrow f(u) = \frac{1}{\Gamma\left(\frac{r_1}{2}\right) 2^{\frac{r_1}{2}}} u^{\frac{r_1}{2}-1} e^{-\frac{u}{2}}, 0 < u < \infty \\ V \sim \chi^2(r_2) &\Rightarrow f(v) = \frac{1}{\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{r_2}{2}}} v^{\frac{r_2}{2}-1} e^{-\frac{v}{2}}, 0 < v < \infty \end{aligned}\right\} \text{indep.}$$

The joint p.d.f. of U and V is then

$$f(u,v) = \begin{cases} \frac{1}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{r_1+r_2}{2}}} u^{\frac{r_1}{2}-1} v^{\frac{r_1}{2}-1} e^{-\frac{(u+v)}{2}} & , 0 < u < \infty, 0 < v < \infty \\ 0 & , \text{o.w.} \end{cases}.$$

Define the new random variables F and Z by writing

$$F = \frac{U/r_1}{V/r_2}, Z = V$$

and we want to find the p.d.f. $g_1(f)$ of F. The equations $f = \frac{u/r1}{v/r_2}$, $z = v$ define a one –to –one transformation that maps

$$A = \{(u,v): 0 < u < \infty, 0 < v < \infty\}$$

onto the set

$$B = \{(f, z): 0 < f < \infty, 0 < z < \infty\}.$$

Now

$$u = \frac{r_1}{r_2} z f,\ v = z \Rightarrow J = \begin{vmatrix} \frac{\partial u}{\partial f} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial f} & \frac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{r_1}{r_2} z & \frac{r_1}{r_2} f \\ 0 & 1 \end{vmatrix} = \frac{r_1}{r_2} z \Rightarrow |J| = \left|\frac{r_1}{r_2} z\right| = \frac{r_1}{r_2} z$$

$$\therefore g(f, z) = f\left(\frac{r_1}{r_2} z f, z\right)|J|$$

$$= \begin{cases} \dfrac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{(r_1+r_2)}{2}}} \left(\frac{r_1}{r_2} z f\right)^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1} e^{-\frac{z}{2}\left(\frac{r_1}{r_2} f+1\right)} \frac{r_1}{r_2} z, & 0 < f < \infty, 0 < z < \infty \\ 0, & \text{o.w.} \end{cases}$$

The marginal p.d.f. of F is then

$$g_1(f) = \int_{\forall z} g(f, z)\,dz$$

$$= \int_0^{\infty} \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} f^{\frac{r_1}{2}-1}}{\Gamma\left(\frac{r_1}{2}\right)\Gamma\left(\frac{r_2}{2}\right) 2^{\frac{(r_1+r_2)}{2}}} z^{\frac{r_1+r_2}{2}-1} e^{-\frac{z}{2}\left(\frac{r_1}{r_2} f+1\right)} dz$$

In this integral let $y = \frac{z}{2}\left(\frac{r_1}{r_2} f + 1\right) \Rightarrow z = \frac{2y}{\frac{r_1}{r_2} f+1}$, $dy = \frac{1}{2}\left(\frac{r_1}{r_2} f + 1\right) dz$